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ALGEBRAIC CRITERIA FOR ASYMPTOTIC STABILITY AT 1:1 RESONANCE†

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The asymptotic stability of a system with two degrees of freedom is investigated in the critical case of two pairs of purely imaginary eigenvalues at 1:1 resonance. It is assumed that the multiple eigenvalues are associated with simple elementary divisors. Algebraic criteria for the asymptotic stability of the full system are constructed; they are based on model equations of the third approximation, on the assumption that the domain of interest is bounded by a certain submanifold of positive measure in the parameter space of the model equations. Certain sufficient conditions for the full system to be unstable are also obtained.

A THEORY of multiple resonance of non-Hamiltonian equations has been developed for reversible systems [1], and also for systems of general form when the elementary divisors belonging to the eigenvalues are not simple [2]; the equilibrium position is, as a rule, unstable. But if the matrix of the linear part of the system is diagonalizable (simple elementary divisors), it has not proved possible to construct stability criteria. It is well known that this problem is transcendental [3]: the surface that separates the classes of asymptotically stable and unstable systems in a 24-dimensional real parameter space is transcendental.

It will be shown below that, despite this transcendence, one can derive algebraic asymptotic stability criteria (the separating surface has algebraic sections).

1. STATEMENT OF THE PROBLEM. CONSTRUCTION OF THE LYAPUNOV FUNCTION

Consider an autonomous system

$$\mathbf{x}^{\bullet} = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(0) = 0, \quad \mathbf{x} \in R^4 \tag{1.1}$$

where $\mathbf{X}(\mathbf{x})$ is a smooth vector field such that the matrix $(\partial\mathbf{X}/\partial\mathbf{x})_0$ has purely imaginary eigenvalues λ_1 and λ_2 satisfying the resonance relation $\lambda_1 = \lambda_2$. Let us assume that λ_1 has simple elementary divisors. The complex normal form of the equations in the third approximation is

$$\begin{aligned} z_1^{\bullet} &= \lambda_1 z_1 + A_{11} z_1^2 \bar{z}_1 + A_{12} z_1 z_2 \bar{z}_2 + A_1 z_1 \bar{z}_1 z_2 + \\ &+ A_2 z_1^2 \bar{z}_2 + A_3 z_2^2 \bar{z}_2 + A_4 z_2^2 \bar{z}_1 \\ z_2^{\bullet} &= \lambda_1 z_2 + A_{21} z_1 \bar{z}_1 z_2 + A_{22} z_2^2 \bar{z}_2 + A_5 z_1^2 \bar{z}_1 + A_6 z_1 z_2 \bar{z}_2 + A_7 z_1^2 \bar{z}_2 + A_8 \bar{z}_1 z_2^2 \end{aligned} \tag{1.2}$$

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$$z_1 = x_1 + ix_2, \quad z_2 = x_3 + ix_4, \quad A_{lm} = a_{lm} + ib_{lm}$$
$$A_m = a_m + ib_m$$

System (1.2) is stable if and only if the system in three variables obtained from (1.2) by changing to polar coordinates r_j, θ_j ($\theta = \theta_2 - \theta_1$ is the resonance angle) is stable

$$\begin{aligned} &r_j^{\bullet} = R_j(r_1, r_2, \theta), \quad \theta^{\bullet} = \Omega(r_1, r_2, \theta), \; j = 1, 2 \\ &(z_j = \sqrt{r_j}\exp(i\theta_j), \quad \bar{z}_j = \sqrt{r_j}\exp(-i\theta_j)) \end{aligned} \tag{1.3}$$

where

$$\begin{aligned} {}^1/_2 R_1 &= a_{11}r_1^2 + a_{12}r_1r_2 + r_1\sqrt{r_1r_2}\,[(a_1 + a_2)\cos\theta + (b_2 - b_1)\sin\theta] + \\ &+ r_2\sqrt{r_1r_2}\,(a_3\cos\theta - b_3\sin\theta) + r_1r_2(a_4\cos 2\theta - b_4\sin 2\theta) \\ {}^1/_2 R_2 &= a_{21}r_1r_2 + a_{22}r_2^2 + r_2\sqrt{r_1r_2}\,[(a_6 + a_8)\cos\theta + (b_6 - b_8)\sin\theta] + \\ &+ r_1\sqrt{r_1r_2}\,(a_5\cos\theta + b_5\sin\theta) + r_1r_2(a_7\cos 2\theta + b_7\sin 2\theta). \end{aligned}$$

We will not write down an explicit formula for the function Ω: suffice it to say that Ω is a trigonometric polynomial of degree 2 whose coefficients are homogeneous functions of degree 1 in r_1 and r_2.

To construct the required Lyapunov function we will use the notion of functional extensions of the integral sheaf of a comparison system. This method was developed in an investigation of gyroscopic systems with complete dissipation, which used a sheaf made up of the total energy and extended cyclic integrals.† It has been used to analyse the stability of gyroscopes with dry friction [4, 5]: the integral sheaf of the system was expanded by adding a certain auxiliary function of the phase variables. It was subsequently shown [6, 7] that the extended complete integral sheaf generates functional extensions of the entire sets of first integrals of the comparison system.

As comparison equations, let us take the model system

$$\begin{aligned} &r_1^{\bullet} = \frac{\partial H}{\partial \theta_1}, \quad r_2^{\bullet} = \frac{\partial H}{\partial \theta_2}, \quad \theta^{\bullet} = \frac{\partial H}{\partial r_1} - \frac{\partial H}{\partial r_2} \\ &H = \lambda_1(r_1 + r_2) + {}^1/_2(b + b_{21} - b_{11})r_1^2 + br_1r_2 + {}^1/_2(b - b_{22} + b_{12}) \times \\ &\times r_2^2 - 2(b_5r_1 + b_3r_2)\sqrt{r_1r_2}\cos\theta + 2(a_5r_1 - a_3r_2) \times \\ &\times \sqrt{r_1r_2}\sin\theta - r_1r_2(b_4\cos 2\theta + a_4\sin 2\theta) \end{aligned} \tag{1.4}$$

which has been studied before [8, 9] and is a special case of Eqs (1.3). Here H is the normal form of the Hamiltonian at multiple resonance. Equations (1.4) can be integrated: the function $\mathbf{W} + C_3$, where

$$\mathbf{W} = C_1[H - \lambda_1(r_1 + r_2)] + C_2(r_1 + r_2)^2, \quad C_j = \text{const}$$

is a complete integral sheaf, since $H = h$, $r_1 + r_2 = c$ are first integrals of the system. We define a functional extension $\mathbf{V} = \mathbf{V}^0(r_1, r_2, \theta, \mathbf{d}) + d_m$ (where $\mathbf{d} = (d_1, \ldots, d_{m-1})$ is a vector of arbitrary constants, $m > 3$) of the integral $\mathbf{W} + C_3$, as follows [7]: $\mathbf{V}$ is a smooth family of functions, of

†MATROSOV B. M., Some questions of the stability of gyroscopic systems. Candidate dissertation, 05.11.03, Kazan, 1959.

which $\mathbf{W}+C_3$ is a special case, i.e. $\mathbf{W}=\mathbf{V}^0|_{\mathbf{p}^{(2)}}$, where $\mathbf{p}^{(2)}=(\varphi_1(C_1,\ C_2),\ \ldots,\ \varphi_{m-1}(C_1,\ C_2))$ is a regular parameterized 2-surface in the space of arbitrary constants $d_1, \ldots, d_{m-1}$.

The expression

$$\begin{aligned} \mathbf{V} &= D_{11}r_1^2 + 2D_{12}r_1r_2 + D_{22}r_2^2 + 2r_1\sqrt{r_1r_2}\,(D_1\cos\theta + \\ &+ D_2\sin\theta) + 2r_2\sqrt{r_1r_2}\,(D_3\cos\theta + D_4\sin\theta) + \\ &+ 2r_1r_2(D_5\cos2\theta + D_6\sin2\theta) + D_7 \end{aligned} \tag{1.5}$$

obviously satisfies this definition ($D_j,\ D_{ij}=\mathrm{const}$).

The derivative of $\mathbf{V}$ along the vector field of Eqs (1.3) is

$$\dot{\mathbf{V}} = r_2^3[\gamma_0 + \sum_{n=1}^{3}(\gamma_{n1}\cos n\theta + \gamma_{n2}\sin n\theta)]$$

$$\gamma_0 = G_0k^3 + G_1k^2 + G_2k + G_3,\quad \gamma_{1m} = 2\sqrt{k}(B_{1m}k^2 + B_{2m}k + B_{3m})$$

$$\gamma_{2m} = 2k(C_{1m}k + C_{2m}),\quad \gamma_{3m} = 2k^{3/2}F_m\quad (m = 1, 2)$$

where $k=r_1/r_2$ is a variable parameter and the coefficients $G_j,\ B_{ij},\ C_{ij}$ and F_m depend linearly on the arbitrary constants D_{ij} and D_j, and the parameters of the problem. Thus, for example

$$G_0 = 4a_{11}D_{11} + 2a_5D_1 + 2b_5D_2,\ G_3 = 4a_{22}D_{22} + 2a_3D_3 - 2b_3D_4$$

(for the other coefficients, see the Appendix). The constants D_{ij} and D_j are chosen so that the coefficients of $\cos2\theta$, $\sin2\theta$, $\cos3\theta$ and $\sin3\theta$ vanish, i.e. we impose upon these numbers the conditions $C_{1m}=C_{2m}=F_m=0$ $(m=1, 2)$. We also require that $B_{2m}=0$, in order to simplify the coefficients of $\cos\theta$ and $\sin\theta$. We have

$$\mathbf{AD} = \mathbf{R},\ \mathbf{D} = (D_{11}, \ldots, D_5)^T \tag{1.6}$$

where $\mathbf{A}$ and $\mathbf{R}$ are 8 by 8 and 8 by 1 matrices, respectively, whose elements are linear functions of the parameters. As the number D_6 appears as a factor on the right-hand side of Eqs (1.6), it does not play an essential role, and we may assume it to be equal to unity. We assume that $\det\mathbf{A}\neq0$. Let $\mathbf{D}=\mathbf{D}^*$ be a family of solutions of Eqs (1.6), depending on the parameters of the problem. Define the Lyapunov function to be $\mathbf{V}^*$, where $\mathbf{V}^*$ is the restriction of $\mathbf{V}$ to this family.

2. CRITERIA FOR $\mathbf{V}^*$ AND $\dot{\mathbf{V}}^*$ TO BE SIGN-DEFINITE

Clearly

$$\dot{\mathbf{V}}^* = r_2^3[\gamma_0^* + \gamma_{11}^*\cos\theta + \gamma_{12}^*\sin\theta]$$

$$\gamma_0^* = G_0^*k^3 + G_1^*k^2 + G_2^*k + G_3^*,\quad \gamma_{1m}^* = 2\sqrt{k}\,(B_{1m}^*k^2 + B_{3m}^*)$$

$$(m = 1, 2)$$

Suppose $G_0^*\neq0$, $G_3^*\neq0$. The function $\dot{\mathbf{V}}^*$ is sign-definite in the cone $r_1\geqslant0$, $r_2\geqslant0$, $0\leqslant\theta<2\pi$ if and only if $\gamma_0^{*2}>\gamma_{11}^{*2}+\gamma_{12}^{*2}$ for any $k>0$ (this inequality remains meaningful even when $k=0$, $k=\infty$, since $\dot{\mathbf{V}}^*$ does not vanish in the planes $r_1=0$, $r_2=0$). Hence, it follows that $\dot{\mathbf{V}}^*$ will be sign-definite in the cone if and only if the equation

$$\gamma_0^{*2} - \gamma_{11}^{*2} - \gamma_{12}^{*2} = 0 \tag{2.1}$$

has no positive roots.

We will now derive a criterion for $\mathbf{V}^*$ to be sign-definite in the domain $r_1 \geqslant 0$, $r_2 \geqslant 0$, $0 \leqslant \theta < 2\pi$, assuming henceforth throughout that $D_{11}^* \neq 0$, $D_{22}^* \neq 0$. We convert $\mathbf{V}^*$ to the form

$$\mathbf{V}^* = r_2^2[\lambda_0 + \lambda_1\cos\theta + \mu_1\sin\theta + \lambda_2\cos 2\theta + \mu_2\sin 2\theta] \tag{2.2}$$

$$\lambda_0 = D_{11}^* k^2 + 2D_{12}^* k + D_{22}^*, \quad \lambda_1 = 2\sqrt{k}(D_1^* k + D_3^*) \tag{2.3}$$

$$\mu_1 = 2\sqrt{k}(D_2^* k + D_4^*), \quad \lambda_2 = 2D_5^* k, \quad \mu_2 = 2k, \; k = r_1/r_2$$

We transform the trigonometric polynomial on the right of (2.2) by substituting $y = \operatorname{tg}(\theta/2)$. This gives

$$\begin{aligned} &\mathbf{V}^* = r_2^2 \Lambda(y)(1+y^2)^{-2} \\ &\Lambda(y) = L_4 y^4 + L_3 y^3 + L_2 y^2 + L_1 y + L_0 \end{aligned} \tag{2.4}$$

$$\begin{aligned} &L_4 = \lambda_0 - \lambda_1 + \lambda_2, \quad L_3 = 2(\mu_1 - 2\mu_2), \quad L_2 = 2(\lambda_0 - 3\lambda_2) \\ &L_1 = 2(\mu_1 + 2\mu_2), \quad L_0 = \lambda_0 + \lambda_1 + \lambda_2 \end{aligned} \tag{2.5}$$

It is obvious that in the singular case $\theta = \pi$, when the above substitution is degenerate, we can calculate $\mathbf{V}^*$ by the formula

$$\mathbf{V}^* = r_2^2 L_4 \tag{2.6}$$

This function vanishes if $L_4 = 0$, but then one of the roots of the polynomial $\Lambda(y)$ goes to infinity. It follows from (2.4) and (2.6) that $\mathbf{V}^*$ is sign-definite in the domain $r_1 \geqslant 0$, $r_2 \geqslant 0$, $0 \leqslant \theta < 2\pi$ if and only if, for any $k > 0$, the polynomial $\Lambda(y)$ has no real roots, including the point at infinity ($\mathbf{V}^*$ does not vanish in the planes $r_1 = 0$, $r_2 = 0$, because $D_{11}^* \neq 0$, $D_{22}^* \neq 0$).

If $\Lambda(y)$ has simple real roots, the function $\mathbf{V}^*$ will change sign. In the case of multiple roots, however, $\mathbf{V}^*$ may retain the same sign. Henceforth we will exclude this situation by stipulating that the discriminant of $\Lambda(y)$ never vanishes at values of k where the polynomial itself has real roots.

3. ASYMPTOTIC STABILITY CRITERIA. INSTABILITY

We will now derive the necessary and sufficient conditions for asymptotic stability, assuming that $\mathbf{V}^*$ is sign-definite in the relevant domain of the parameter space.

Let $\mathbf{A}$ be the matrix of the linear system (1.6), D_{ij}^*, D_j^* parameters of $\mathbf{V}$ that satisfy Eqs (1.6), G_j^*, B_{ij}^* the corresponding values of the coefficients in the derivative $\mathbf{V}^{\cdot *}$, and L_j are calculated from formulae (2.3) and (2.5). We put $k = r_1/r_2$, and assume $F(k)$ is the discriminant of the polynomial $\Lambda(y)$, and π is the set of positive values of k for which the polynomial itself has real roots. The condition $F|_\pi \neq 0$ (or $L_3|_{L_4=0} \neq 0$ for the point at infinity) guarantees that $\Lambda(y)$ will have no multiple roots. We then have the following theorem.

Theorem 1. Let $\det\mathbf{A} \neq 0$, $G_0^* \neq 0$, $G_3^* \neq 0$, $D_{11}^* \neq 0$, $D_{22}^* \neq 0$, $F|_\pi \neq 0$, $L_3|_{L_4=0} \neq 0$, $G_0^* D_{11}^* < 0$ and suppose that the real algebraic equation (2.1) has no positive roots. The equilibrium position of the complete system (1.1) is asymptotically stable if, for any $k > 0$, the polynomial $\Lambda(y)$ has no real roots, including the point at infinity. Otherwise, if this polynomial has at least one real root for some $k > 0$, the equilibrium is unstable.

Proof. The assumptions of the theorem imply that $\mathbf{V}^{\cdot *}$ is sign-definite in the domain $r_1 \geqslant 0$, $r_2 \geqslant 0$, $0 \leqslant \theta < 2\pi$ (the higher-order terms dropped when deriving the model equations (1.2) do not affect the sign of $\mathbf{V}^{\cdot *}$, since $\mathbf{V}^*$ and the right-hand sides of Eqs (1.2) are homogeneous polynomials in z_j, $\bar{z}_j$). Obviously, $\operatorname{sign}\mathbf{V}^{\cdot *} = \operatorname{sign}G_0^*$. If $\Lambda(y)$ has no real roots, then $\mathbf{V}^*$ is also sign-definite in that domain and $\operatorname{sign}\mathbf{V}^* = \operatorname{sign} D_{11}^*$. It follows from the condition $G_0^* D_{11}^* < 0$ that $\mathbf{V}^* \mathbf{V}^* < 0$, and therefore $\mathbf{V}^*$ satisfies all the conditions of Lyapunov's theorem of asymptotic stability. Otherwise, if $\Lambda(y)$ has a real root for some $k > 0$, then $\mathbf{V}^*$ changes sign, so that the conditions of Lyapunov's instability theorem are satisfied.

Theorem 1 clearly furnishes algebraic criteria for the stability of the complete system in the domain $\mathbf{V}^{\cdot *} > 0$, since the conditions for the polynomial $\Lambda(y)$ to have no real roots for any $k > 0$ are algebraic [10, p. 249].

Let us assume that in the domain in which $\mathbf{V}^{\cdot *}$ is sign-definite the parameters G_0^*, D_{11}^* satisfy the condition $G_0^* D_{11}^* > 0$. This means that $\mathbf{V}^*$, varying in the neighbourhood of $r_1 = r_2 = 0$, assumes values of the same sign as $\mathbf{V}^{\cdot *}$. Hence, the equilibrium position is unstable by Lyapunov's instability condition. We have thus proved the following theorem.

Theorem 2. Let $\det\mathbf{A} \neq 0$, $G_0^* \neq 0$, $G_3^* \neq 0$, $D_{11}^* \neq 0$, $D_{22}^* \neq 0$, $G_0^* D_{11}^* > 0$ and assume that Eq. (2.1) has no positive roots. Then the equilibrium position of the complete system (1.1) is unstable.

It is clear that these last two theorems answer all the questions as to the stability of system (1.1) in the domain $\mathbf{V}^{\cdot *} > 0$.

To obtain further information, we consider an additional function $\mathbf{V}^{**}$, which differs from $\mathbf{V}^*$ in the values of the constants D_{ij} and D_j. To define it, we simplify the derivative $\mathbf{V}^{\cdot}$, equating the coefficients of $\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta$ to zero: $B_{km} = 0$, $F_m = 0$ $(m = 1, 2;\ k = 1, 2, 3)$. This gives a set of linear equations analogous to (1.6)

$$\mathbf{BD} = \mathbf{S} \tag{3.1}$$

Here $\mathbf{D}$ has the same meaning as before and $\mathbf{B}$ and $\mathbf{S}$ are 8 by 8 and 8 by 1 matrices, respectively, with elements that are linear functions of the parameters, $\mathbf{B} \neq \mathbf{A}$.

Assuming that $\det\mathbf{B} \neq 0$, let $\mathbf{V}^{**}$ be the restriction of $\mathbf{V}$ to the family of solutions of Eqs (3.1). The derivative $\mathbf{V}^{\cdot **}$ is sign-definite if and only if the equation

$$(G_0^{**}k^3 + G_1^{**}k^2 + G_2^{**}k + G_3^{**})^2 - 4k^2[(C_{11}^{**}k + C_{21}^{**})^2 + (C_{12}^{**}k + C_{22}^{**})^2] = 0$$

has no positive roots. Repeating the previous arguments, we obtain another two theorems, analogous to Theorems 1 and 2.

Note that if the function (1.5) is expressed in terms of the original complex variables, it becomes a homogeneous polynomial of degree four which is invariant under the transformation $\mathbf{z} \to \mathbf{z}\exp(i\alpha)$, where $\mathbf{z} = (z_1, z_2)$ and α is a parameter. With the Lyapunov function constructed as a homogeneous quadratic polynomial, some necessary and sufficient conditions have been established for stability in the case of multiple resonance, as well as a few sufficient conditions.†

4. CRITERIA FOR THE EXISTENCE OF INVARIANT RAYS. LYAPUNOV INSTABILITY

As we know, solutions asymptotic to zero play an important role in stability analyses. We shall now establish some conditions for the existence of invariant rays of system (1.3), i.e. particular solutions of the form

$$r_1 = kr_2, \quad \theta = \theta^* \tag{4.1}$$

†KHAZINA G. G. and KHAZIN L. G., On the possibility of resonance stabilization of a system of oscillators. Preprint No. 130, Inst. Prikl. Mat. Akad. Nauk SSSR, Moscow, 1978.

where k and θ^* are constants. To that end, we substitute (4.1) into Eqs (1.3) and, introducing the variable $z = \exp(i\theta)$, reduce the resulting equations to the form

$$
\begin{aligned}
f_1 &\equiv v_0 + v_1 z + v_2 z^2 + \bar{v}_1 z^3 + \bar{v}_0 z^4 = 0 \\
f_2 &\equiv w_0 + w_1 z + w_2 z^2 + \bar{w}_1 z^3 + \bar{w}_0 z^4 = 0
\end{aligned}
\tag{4.2}
$$

where

$$
v_0 = \tfrac{1}{2}[(a_4 - ka_7) - i(b_4 + kb_7)], \quad v_2 = (a_{11} - a_{21})k + (a_{12} - a_{22})
$$

$$
v_1 = \frac{1}{2\sqrt{k}} \{[-a_5 k^2 + (a_1 + a_2 - a_6 - a_8)k + a_3] - i[b_5 k^2 + (b_6 - b_8 + b_1 - b_2)k + b_3]\}
$$

The coefficients w_0, w_1 and w_2 have exactly the same form, except for the substitutions $a_j \to -b_j$, $b_j \to a_j$, $a_{lm} \to -b_{lm}$.

The equality

$$
\mathbf{R} = 0 \tag{4.3}
$$

where $\mathbf{R}$ is the resultant of the polynomials f_1 and f_2, yields necessary and sufficient conditions for system (4.2) to be consistent. Obviously, $\mathbf{R} = \bar{\mathbf{R}}$. The resultant reduces to a real polynomial in k, of degree 14. Thus, the computation of k reduces to finding the positive roots of Eq. (4.3). For these values of k system (4.2) has solutions that belong to the unit circle if and only if the greatest common divisor of f_1 and f_2

$$
g_0(k) + g_1(k)z + \ldots + g_m(k)z^m \quad (m \leqslant 4) \tag{4.4}
$$

is non-trivial and has zeros $\zeta_l = \exp(i\theta_l)$. The polynomial (4.4) is constructed by using Euclid's algorithm based on division of polynomials with a remainder. We have thus proved the following lemma.

Lemma. Equations (1.3) have a particular solution of the form $r_1 = kr_2$, $\theta = \theta^*$ if and only if k is a positive root of Eq. (4.3) and $\zeta = \exp(i\theta^*)$ is a root of the polynomial (4.4).

We can now establish conditions for system (1.3) to be unstable. To that end we calculate the derivative r_2 along an invariant ray $r_1 = kr_2$, $\theta = \theta^*$

$$
r_2^{\cdot} = r_2^2 R_2(k, 1, \theta^*)
$$

Obviously, the trivial solution is unstable if $R_2(k, 1, \theta^*) > 0$. The instability is retained by the full system [11]. We have thus proved the following theorem.

Theorem 3. If Eqs (1.3) have a family of particular solutions $r_1 = k_j r_2$, $\theta = \theta_l^*$ and at the same time $R_2(k_j, 1, \theta_l^*) > 0$ for some values of the parameters k and θ^* in that family, then system (1.1) is unstable in Lyapunov's sense.

APPENDIX

The expressions for the coefficients of $\mathbf{V}^*$ are

$$
\begin{aligned}
G_1 &= 4a_{12}D_{11} + 4(a_{11} + a_{21})D_{12} + 2(2a_1 + a_2 + a_6)D_1 + \\
&+ 2(b_2 + b_6 - 2b_1)D_2 + 4a_5 D_3 + 4b_5 D_4 + 4a_7 D_5 + 4b_7 D_6 \\
G_2 &= 4(a_{12} + a_{22})D_{12} + 4a_{21}D_{22} + 4a_3 D_1 - 4b_3 D_2 + \\
&+ 2(a_1 + 2a_6 + a_8)D_3 + 2(2b_6 - b_1 - b_8)D_4 + 4a_4 D_5 - 4b_4 D_6
\end{aligned}
$$

$$B_{11} = 2(a_1 + a_2)D_{11} + 2a_5D_{12} + (3a_{11} + a_{21} + a_7)D_1 + (b_{21} - b_{11} + b_7)D_2 + 2a_3D_5 + 2b_5D_6$$

$$B_{21} = 2a_3D_{11} + 2(a_1 + c_3 + a_6 + a_8)D_{12} + 2a_5D_{22} + (3a_{12} + a_{22} + 2a_4)D_1 +$$
$$+ (b_{22} - b_{12} - 2b_4)D_2 + (a_{11} + 2a_7 + 3a_{21})D_3 + (b_{21} - b_{11} + 2b_7)D_4 + 2(a_1 + a_6)D_5 +$$
$$+ 2(b_6 - b_1)D_6$$

$$B_{31} = 2a_3D_{12} + 2(a_6 + a_8)D_{22} + (3a_{22} + a_{12} + a_4)D_3 +$$
$$+ (b_{22} - b_{12} - b_4)D_4 + 2a_3D_5 - 2b_3D_6$$

$$B_{12} = 2(b_2 - b_1)D_{11} + 2b_5D_{12} + (b_{11} - b_{21} + b_7)D_1 +$$
$$+ (3a_{11} + a_{21} - a_7)D_3 - 2b_5D_5 + 2a_5D_6$$

$$B_{22} = -2b_3D_{11} + 2(b_2 + b_6 - b_1 - b_8)D_{12} + 2b_5D_{22} + (b_{12} - b_{22} - 2b_4)D_1 +$$
$$+ (3a_{12} + a_{22} - 2a_4)D_2 + (b_{11} - b_{21} + 2b_7)D_3 + (a_{11} + 3a_{21} - 2a_7)D_4 +$$
$$+ 2(b_1 - b_6)D_5 + 2(a_1 + a_6)D_6$$

$$B_{32} = -2b_3D_{12} + 2(b_6 - b_8)D_{22} + (b_{12} - b_{22} - b_4)D_3 + (a_{12} + 3a_{22} - a_4)D_4 +$$
$$+ 2b_3D_5 + 2a_3D_6$$

$$C_{11} = 2a_4D_{11} + 2a_7D_{12} + (a_1 + a_8 + 2a_2)D_1 + (b_1 - 2b_2 + b_8)D_2 + a_5D_3 - b_5D_4 +$$
$$+ 2(a_{11} + a_{21})D_5 + 2(b_{21} - b_{11})D_6$$

$$C_{21} = 2a_4D_{12} + 2a_7D_{22} + a_3D_1 + b_3D_2 + (a_2 + a_6 + 2a_8)D_3 + (2b_8 - b_2 - b_6)D_4 +$$
$$+ 2(a_{12} + a_{22})D_5 + 2(b_{22} - b_{12})D_6$$

$$C_{12} = -2b_4D_{11} + 2b_7D_{12} + (2b_2 - b_1 - b_8)D_1 + (a_1 + 2a_2 + a_8)D_2 + b_5D_3 +$$
$$+ a_5D_4 + 2(b_{11} - b_{21})D_5 + 2(a_{11} + a_{21})D_6$$

$$C_{22} = -2b_4D_{12} + 2b_7D_{22} - b_3D_1 + a_3D_2 + (b_2 + b_6 - 2b_8)D_3 +$$
$$+ (a_2 + a_6 + 2a_8)D_4 + 2(b_{12} - b_{22})D_5 + 2(a_{12} + a_{22})D_6$$

$$F_1 = a_4D_1 + b_4D_2 + a_7D_3 - b_7D_4 + 2(a_2 + a_8)D_5 + 2(b_8 - b_2)D_6$$
$$F_2 = -b_4D_1 + a_4D_2 + b_7D_3 + a_7D_4 + 2(b_2 - b_8)D_5 + 2(a_2 + a_8)D_6$$

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REFERENCES

1. KRASIL'NIKOV P. S. and TKHAI V. N., Reversible systems. Stability at 1:1 resonance. *Prikl. Mat. Mekh.* **56**, 4, 570–579, 1992.
2. KHAZIN L. G. and SHNOL' E. E., Stability of critical equilibrium positions. Scientific Centre for Biological Research, Akad. Nauk SSSR, Pushchino, 1985.
3. KHAZIN L. G. and SHNOL' E. E., The simplest cases of algebraic unsolvability in asymptotic stability problems. *Dokl. Akad. Nauk SSSR* **240**, 6, 1309–1311, 1978.
4. KREMENTULO V. V., Investigation of the stability of a gyroscope taking dry friction on the axis of the internal Cardan ring (gimbal) into account. *Prikl. Mat. Mekh.* **23**, 5, 968–970, 1959.
5. KREMENTULO V. V., Stability of a gyroscope in which the axis of the external gimbal is vertical, taking into account dry friction in the suspension axes. *Prikl. Mat. Mekh.* **24**, 3, 568–571, 1960.
6. KRASIL'NIKOV P. S., The construction of certain function spaces and their relationship to Lyapunov's direct method. In *Analytical and Numerical Methods of Investigating Mechanical Systems*, pp. 12–17. Mosk. Aviats. Inst., Moscow, 1989.
7. KRASIL'NIKOV P. S., Generalized spaces of germs of smooth solutions of a first-order equation and their relationship to Lyapunov's direct method. *Izv. Vuzov, Mat.* 5, 47-53, 1990.
8. SOKOL'SKII A. G., On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies. *Prikl. Mat. Mekh.* **38**, 5, 791–799, 1974.
9. KOVALEV A. M. and CHUDNENKO A. N., On the stability of equilibrium positions of a two-dimensional Hamiltonian system in the case of equal frequencies. *Dokl. Akad. Nauk UkrSSR, Ser. A* 11, 1010–1013, 1977.

10. ROUTH E. J., *Dynamics of a System of Rigid Bodies,* Vol. 2. Nauka, Moscow, 1983.
11. FURTA S. D., On the asymptotic solutions of systems of differential equations in the case of purely imaginary eigenvalues. *Differents. Uravn.* **26**, 8, 1348–1351, 1990.

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